Polynomial Properties

The use of polynomials plays an important role in systems and control theory. The Laplace transform of many signals results in a rational function, with a polynomial divided by a polynomial. The polynomial in the denominator determines a great deal about the time response of the system, including stability. For testing the stability of continuous-time polynomials, we have the Routh-Hurwitz criterion, for discrete-time polynomials, we have the Jury-Raible test. To obtain the time response of a function expressed in the Laplace transform variable (or the $\mathcal{Z}$-transform variable), we find it most helpful to expand the function using Partial Fraction Expansions. This allows us to get the function into terms that enable us to use a table of transform pairs. The use of polynomials is a field in itself, and so we introduce important terminology in Section C. Section D discusses an important polynomial equation called the Diophantine Equation. This equation often arises in the design of control systems.

A. The Routh-Hurwitz Criterion

Given the polynomial, $D(s)$, which may be the denominator of a transfer function,

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_o$$

we seek to find when the roots of this polynomial lie in the Left Half Plane (LHP). This approach relies on the use of an array, which may be computed as follows:

$$\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
s^2 & k_1 & k_2 & & & \\
s & l_1 & & & & \\
1 & m_1 & & & &
\end{array}$$

where

$$b_1 = \frac{-1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}, \quad b_2 = \frac{-1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}, \quad b_3 = \frac{-1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-6} \\ a_{n-1} & a_{n-7} \end{vmatrix}, \quad \cdots$$

$$c_1 = \frac{-1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}, \quad c_2 = \frac{-1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}, \quad c_3 = \frac{-1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-7} \\ b_1 & b_4 \end{vmatrix}, \quad \cdots$$

$$\vdots$$

$$m_1 = \frac{-1}{l_1}\begin{vmatrix} k_1 & k_2 \\ l_1 & 0 \end{vmatrix} = k_2.$$

> **The Routh-Hurwitz Criterion:** The number of roots in the Right Half Plane is equal to the number of sign changes in the first *column* of the Routh Array.

The Routh-Hurwitz Criterion provides necessary and sufficient conditions for the roots of a polynomial to lie in the RHP. The first two rows in the Routh array come directly from the polynomial. The elements in the other rows are obtained from computing the determinants based on the previous two rows. This test involves simply testing only the signs of the elements in the first column. If all the elements are positive (or all negative) then the polynomial is *Hurwitz* (i.e. has all its roots in the LHP).

<u>Example A.1</u>: $D(s) = s^3 + s^2 + 2s + 8$.

Forming the Routh array:

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 8 \\ s & \alpha_1 & 0 \\ 1 & \beta_1 & \end{array} \qquad \alpha_1 = \frac{-1}{1}\begin{vmatrix} 1 & 2 \\ 1 & 8 \end{vmatrix} = -(8 - 2^2) = -6$$

$$\beta_1 = \frac{-1}{\alpha_1}\begin{vmatrix} 1 & 8 \\ \alpha_1 & 0 \end{vmatrix} = -(-8\alpha_1) / \alpha_1 = 8$$

and the Routh Array becomes

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 8 \\ s & -6 & \\ 1 & 8 & \end{array} \Rightarrow \begin{array}{c|c} s^3 & + \\ s^2 & + \\ s & - \\ 1 & + \end{array} \begin{array}{c} \\ \\ 1 \\ 2 \end{array} \Rightarrow \text{2 sign changes}$$

Since there are two sign changes, we know that there are two roots in the RHP. We could have seen this by factoring $D(s) = (s+2)(s^2 - s + 4)$.

<u>Example A.2</u>: The Routh-Hurwitz criterion may also be used to find parameters that ensure that the polynomial is Hurwitz. Suppose we have: $D(s) = s^3 + (1-K)s^2 + 2(1-K)s + 8$. We would like to know which values of K make this polynomial Hurwitz. We form the array

$$\begin{array}{c|cc} s^3 & 1 & 2(1-K) \\ s^2 & 1-K & 8 \\ s & \alpha_1 & 0 \\ 1 & \beta_1 & \end{array} \qquad \alpha_1 = \frac{-1}{1-K}\begin{vmatrix} 1 & 2(1-K) \\ 1-K & 8 \end{vmatrix} = \frac{-6-4K+2K^2}{1-K}$$

$$\beta_1 = \frac{-1}{\alpha_1}\begin{vmatrix} 1-K & 8 \\ \alpha_1 & 0 \end{vmatrix} = -(-8\alpha_1)/\alpha_1 = 8$$

Thus, in order for the polynomial to be Hurwitz, we must have:

i) $1-K > 0 \quad \Rightarrow \quad 1 > K$

ii) $\dfrac{-6-4K+2K^2}{1-K} > 0 \quad \Rightarrow \quad 0 < K^2 - 2K - 3 = (1-K)^2 - 4$.

$\Rightarrow \quad -1 > K$

Combining these two criteria, we see that we must have: $K < -1$. Indeed, this may be verified by noting that $D(s) = (s+2)(s^2 - (K+1)s + 4)$ which is Hurwitz if and only if $K + 1 < 0$.

The only time we have problems using the Routh-Hurwitz criterion as stated is in the event that we have to divide by zero. This can only happen in the following two cases

A. There is a zero element in the first column

B. There is an entire row of zeros

In the first case, we would replace the zero element by a "small" number (e.g. ε, γ, δ, ρ, etc.). We would continue to calculate the array with the small symbol appearing as a variable. In order to evaluate the array, we let $\varepsilon \to 0$, $\delta \to 0$, etc. and determine the signs on the first column (the small numbers may be considered either positive or negative – it doesn't affect the number of sign changes). The number of RHP poles equals the number of sign changes as before.

<u>Example A.3</u>: $D(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

Routh Array

$$
\begin{array}{c|ccc}
s^5 & 1 & 2 & 11 \\
s^4 & 2 & 4 & 10 \\
s^3 & 0 \to \varepsilon & 6 & \\
s^2 & (4\varepsilon - 12)/\varepsilon \approx -12/\varepsilon & 10 & \\
s & 6 & & \\
1 & 10 & &
\end{array}
$$

where

$$
b_1 = \frac{-1}{2}\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0, \quad b_2 = \frac{-1}{2}\begin{vmatrix} 1 & 11 \\ 2 & 10 \end{vmatrix} = 6,
$$

$$
c_1 = \frac{-1}{\varepsilon}\begin{vmatrix} 2 & 4 \\ \varepsilon & 6 \end{vmatrix} = \frac{4\varepsilon - 12}{\varepsilon}, \quad c_2 = \frac{-1}{\varepsilon}\begin{vmatrix} 2 & 10 \\ \varepsilon & 0 \end{vmatrix} = 10
$$

$$
d_1 \approx \frac{\varepsilon}{12}\begin{vmatrix} \varepsilon & 6 \\ -12/\varepsilon & 10 \end{vmatrix} = \frac{\varepsilon(10\varepsilon + 6(12/\varepsilon))}{12} \approx 6, \qquad e_1 = \frac{-1}{6}\begin{vmatrix} d_1 & 10 \\ 6 & 0 \end{vmatrix} = 10
$$

where approximations have been made since ε is small. Evaluating the number of sign changes, we see that

$$
\begin{array}{c|c}
 & \varepsilon > 0 \\
s^5 & + \\
s^4 & + \\
s^3 & + \\
s^2 & - \\
s & + \\
1 & +
\end{array}
\qquad\qquad
\begin{array}{c|c}
 & \varepsilon < 0 \\
s^5 & + \\
s^4 & + \\
s^3 & - \\
s^2 & + \\
s & + \\
1 & +
\end{array}
$$

Note that the number of sign changes doesn't depend on the sign of ε. In this case, there are two sign changes, and thus, there are two roots in the RHP and three roots in the LHP.

The second problem is caused by a row of zeros. In this case we use the following fact:

Fact: *When a polynomial has an even polynomial as a factor, the Routh Array will have a row of zeros. The row above the row of zeros contains the coefficients of the even polynomial.*

Even Polynomial: a polynomial whose exponents are multiples of 2.

$$\text{e.g.}\quad s^2+2,\quad s^4-2s^2+1,\quad s^6+2s^4+s^2-7$$

The even polynomial that appears in the Routh-Hurwitz array is called the *Auxiliary Polynomial*. There are two possible ways of dealing with the even polynomial:

A. Factor the polynomial out, and deal with it separately

B. Differentiate the even polynomial

The second option may not seem, at first, to be a useful approach, but it actually provides useful information

<u>Example A.4</u>: $D(s) = s^3 + s^2 + 2s + 2$

Forming the Routh Array gives:

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s & 0 & \\ 1 & ? & \end{array}$$

From this, we know that the auxiliary polynomial is given by

$$Q_a(s) = (1)s^2 + (2)s^o = s^2 + 2.$$

Thus, we know that $D(s) = (s^2+2)(s+?)$. Indeed, by synthetic (polynomial) division, it can be shown that

$$D(s) = (s^2+2)(s+1) = (s+j\sqrt{2})(s-j\sqrt{2})(s+1).$$

If the even polynomial is too complex, it may not be very easy to divide by. In this case we may wish to use the second approach: Fill in the row of zeros with the derivative of the even polynomial.

<u>Example A.5</u>: $D(s) = 6s^5 + 3s^4 + 4s^3 + 2s^2 + 2s + 1$

Routh Array:

$$\begin{array}{c|ccc} s^5 & 6 & 4 & 2 \\ s^4 & 3 & 2 & 1 \\ s^3 & 0 & 0 & 0 \\ s^2 & ? & ? & ? \end{array}$$

Auxiliary Polynomial: $Q_a(s) = 3s^4 + 2s^2 + 1 \quad \Rightarrow \quad \dfrac{dQ_a(s)}{ds} = 12s^3 + 4s$

Thus, we fill in the Routh Array

$$\begin{array}{c|ccc} s^5 & 6 & 4 & 2 \\ s^4 & 3 & 2 & 1 \\ s^3 & \mathbf{12} & \mathbf{4} & \\ s^2 & 1 & 1 & \\ s & -8 & & \\ 1 & 1 & & \end{array}$$

<u>Example A.6</u>:

$$D(s) = s^4 + 4 = (s+1+j)(s+1-j)(s-1+j)(s-1-j) \quad \Rightarrow \quad \begin{array}{c|ccc} s^4 & 1 & 0 & 4 \\ s^3 & 0 & 0 & \end{array}$$

$$\Rightarrow \quad Q_a(s) = s^4 + 4, \qquad \Rightarrow \qquad \frac{dQ_a}{ds} = 4s^3$$

$$\begin{array}{c|ccc} s^4 & 1 & 0 & 4 \\ s^3 & \mathbf{4} & \mathbf{0} & \\ s^2 & 0 \to \varepsilon & 4 & \\ s & \approx \frac{-16}{\varepsilon} & & \\ 1 & 4 & & \end{array}$$

Thus, since there are two sign changes (for any sign of e), there must be two unstable poles, which there are!

Before continuing further, we step back and look at properties of even polynomials

A.1. Even Polynomials

Consider the general 4th order even polynomial:

$$D(s) = s^4 + as^2 + b.$$

If we let $x = s^2$, then $D(x) = x^2 + ax + b$ which is quadratic expression. Since the roots of the quadratic always appear either in complex conjugate pairs (or as real values), then we may note that if we have complex roots, then we have roots at $x = r^2 e^{\pm j\theta}$. In this case, we have $s = \pm x^{1/2} = \pm\, r\, e^{\pm j\theta/2}$, which consists of four roots. The ± sign before the radical implies that the roots will be symmetric with respect to the imaginary axis, the ± sign in the exponent implies that the roots will also be symmetric with respect to the real axis. Thus, the roots appear in a sort of rectangular configuration.

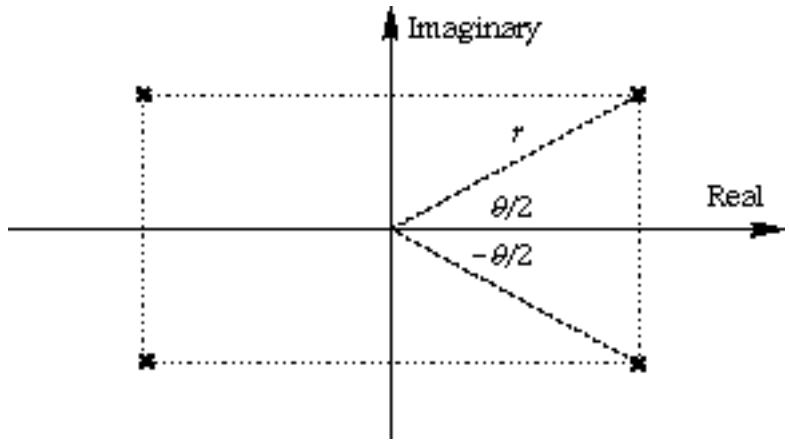

Figure A.1 Poles in the Complex Plane

If the roots are real, then $D(x) = x^2 + ax + b = (x+\alpha)(x+\beta) \Rightarrow D(s) = (s^2+\alpha)(s^2+\beta)$, giving two even polynomials. A 2nd order even polynomial will have roots either both on the imaginary axis or both on the real axis.

B. Partial Fraction Expansions

It is easy to verify that

$$\frac{2}{s^2-1} = \frac{1}{s-1} + \frac{-1}{s+1}$$

The expression on the right side of this equation is called the *partial fraction expansion* of $2/(s^2-1)$. We review the rules needed to perform these expansions.

We will assume throughout that $D(s)$ is a monic polynomial of order n, i.e.

$$\begin{aligned} D(s) &= s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_o \\ &= (s+p_1)^{r_1}(s+p_2)^{r_2}\cdots(s+p_q)^{r_q}(s^2+2\zeta_1\omega_1 s+\omega_1^2)^{k_1}\cdots(s^2+2\zeta_l\omega_l s+\omega_l^2)^{k_l} \end{aligned}$$

where $r_1 + r_2 + \cdots + r_q + 2(k_1 + \cdots + k_l) = n$, and $\omega_i > 0, |\zeta_i| < 1$. We also assume that $N(s)$ is an mth order polynomial

$$N(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_o .$$

It is theoretically possible to write any rational expression, $N(s)/D(s)$, as a sum of rational expressions whose denominators involve powers of polynomials of degree not greater than one or two. If $N(s)$ and $D(s)$ are polynomials, and *the degree of* $N(s)$ *is less than the degree of* $D(s)$, then it follows from complex variable theory that

$$\frac{N(s)}{D(s)} = G_1(s) + G_2(s) + \cdots + G_N(s) \tag{A.1}$$

where each $G_i(s)$ has one of the forms

$$\frac{A}{(s + p_i)^j} \quad \text{or} \quad \frac{Bs + C}{(s^2 + 2\zeta_i \omega_i s + \omega_i^2)^j}$$

for some integer *j*, and where the term $s^2 + 2\zeta_i \omega_i s + \omega_i^2$ is irreducible, in the sense that it has complex roots. The right hand side of (A.1) is called the partial fraction expansion of $N(s)/D(s)$ and each $G_i(s)$ is called a partial fraction. We will not prove this algebraic result, but will instead, give rules for obtaining the decomposition.

To find the partial fraction decomposition of a rational expression, $N(s)/D(s)$, it is <u>*essential*</u> that $N(s)$ have lower degree than $D(s)$ (i.e. $n > m$). If this is not the case, then long division should be employed to arrive at such an expression. For example, given,

$$\frac{N(s)}{D(s)} = \frac{s^3 - 6s^2 + 5s - 3}{s^2 - 1}$$

we obtain, by long division (sometimes called synthetic division),

$$\frac{N(s)}{D(s)} = \frac{s^3 - 6s^2 + 5s - 3}{s^2 - 1} = s - 6 + \frac{6s - 9}{s^2 - 1} .$$

The partial fraction expansion is then found for the term $(6s - 9)/(s^2 - 1)$.

In order to obtain the expansion (A.1), we begin by expressing the denominator $D(s)$, as a product of the factors $(s + p_i)$ or irreducible quadratic factors $s^2 + 2\zeta_i \omega_i s + \omega_i^2$. Repeated

factors are then collected so that $D(s)$ is a product of different factors of the form $(s+p_i)^j$ or $(s^2+2\zeta_i\omega_i s+\omega_i^2)^j$, where j is a positive integer. We may then apply the following rules.

Rule 1. *For each factor of the form* $(s+p_i)^j$, *where* $j \geq 1$, *the expansion* (A.1) *contains a sum of* j *partial fractions of the form*

$$\frac{A_{i1}}{s+p_i}+\frac{A_{i2}}{(s+p_i)^2}+\cdots+\frac{A_{ij}}{(s+p_i)^j}$$

where each A_{ik} *is a real number, and they are given by the formulae*

$$A_{ij}=(s+p_i)^j G(s)\Big|_{s=-p_i}$$

$$A_{i(j-1)}=\frac{d}{ds}\left[(s+p_i)^j G(s)\right]\Big|_{s=-p_i}$$

$$\vdots$$

$$A_{i1}=\frac{1}{(j-1)!}\frac{d^{j-1}}{ds^{j-1}}\left[(s+p_i)^j G(s)\right]\Big|_{s=-p_i}$$

Rule 2. *For each factor of the form* $(s^2+2\zeta_i\omega_i s+\omega_i^2)^j$, *where* $j \geq 1$, *the expansion* (A.1) *contains a sum of* j *partial fractions of the form*

$$\frac{B_{i1}s+C_{i1}}{s^2+2\zeta_i\omega_i s+\omega_i^2}+\frac{B_{i2}s+C_{i2}}{(s^2+2\zeta_i\omega_i s+\omega_i^2)^2}+\cdots+\frac{B_{ij}s+C_{ij}}{(s^2+2\zeta_i\omega_i s+\omega_i^2)^j}$$

where, for each k, B_{ik} *and* C_{ik} *are real numbers.*

Note that Rule 2 gives us the form for the expansion of the irreducible terms, but does not give the explicit formulae for computing the coefficients. There are several ways in which to deal with these terms, some being easier than others.

<u>Example A.7</u>: $\dfrac{N(s)}{D(s)}=\dfrac{s+1}{(s+2)^3}$. We know that this may be written in the form

$$\frac{N(s)}{D(s)}=\frac{s+1}{(s+2)^3}=\frac{A_1}{s+2}+\frac{A_2}{(s+2)^2}+\frac{A_3}{(s+2)^3}$$

where we must now solve for the constants A_1, A_2, and A_3. Using the formulae given above, we see that

$$A_3 = (s+2)^3 \left.\frac{s+1}{(s+2)^3}\right|_{s=-2} = (s+1)\Big|_{s=-2} = -1$$

$$A_2 = \tfrac{d}{ds}(s+2)^3 \left.\frac{s+1}{(s+2)^3}\right|_{s=-2} = \tfrac{d}{ds}(s+1)\Big|_{s=-2} = 1\Big|_{s=-2} = 1$$

$$A_1 = \left(\tfrac{1}{2!}\right)\frac{d^2}{ds^2}\left[(s+2)^3 \frac{s+1}{(s+2)^3}\right]\Bigg|_{s=-2} = \left(\tfrac{1}{2!}\right)\frac{d^2}{ds^2}(s+1)\Big|_{s=-2} = 0\Big|_{s=-2} = 0$$

and we may verify that

$$\frac{N(s)}{D(s)} = \frac{s+1}{(s+2)^3} = \frac{1}{(s+2)^2} - \frac{1}{(s+2)^3}.$$

<u>Example A.8</u>: $\dfrac{N(s)}{D(s)} = \dfrac{s^2+10s+4}{(s+2)^3(s+3)}$. We know that this may be written in the form

$$\frac{N(s)}{D(s)} = \frac{s^2+10s+4}{(s+2)^3(s+3)} = \frac{A_1}{s+2} + \frac{A_2}{(s+2)^2} + \frac{A_3}{(s+2)^3} + \frac{A_4}{s+3}.$$

Applying the above rules, we obtain

$$A_3 = (s+2)^3 \left.\frac{s^2+10s+4}{(s+2)^3(s+3)}\right|_{s=-2} = \left.\frac{s^2+10s+4}{(s+3)}\right|_{s=-2} = \frac{4-20+4}{(1)} = -12$$

$$A_2 = \tfrac{d}{ds}\left[(s+2)^3 \frac{s^2+10s+4}{(s+2)^3(s+3)}\right]\Bigg|_{s=-2} = \tfrac{d}{ds}\left[\frac{s^2+10s+4}{(s+3)}\right]\Bigg|_{s=-2} = \left.\frac{s^2+6s+26}{(s+3)^2}\right|_{s=-2}$$

$$= \frac{4-12+26}{(1)} = 18$$

$$A_1 = \left(\tfrac{1}{2!}\right)\frac{d^2}{ds^2}\left[\frac{s^2+10s+4}{(s+3)}\right]\Bigg|_{s=-2} = \left(\tfrac{1}{2!}\right)\tfrac{d}{ds}\left[\frac{s^2+6s+26}{(s+3)^2}\right]\Bigg|_{s=-2} = \left(\tfrac{1}{2!}\right)\left.\frac{-34}{(s+3)^3}\right|_{s=-2} = -17$$

$$A_4 = (s+3)\left.\frac{s^2+10s+4}{(s+2)^3(s+3)}\right|_{s=-3} = \left.\frac{s^2+10s+4}{(s+2)^3}\right|_{s=-3} = \frac{9-30+4}{(-1)} = 17.$$

Thus,

$$\frac{N(s)}{D(s)} = \frac{s^2+10s+4}{(s+2)^3(s+3)} = \frac{-17}{s+2} + \frac{18}{(s+2)^2} + \frac{-12}{(s+2)^3} + \frac{17}{s+3}.$$

<u>Example A.9</u>: $\dfrac{N(s)}{D(s)} = \dfrac{s^2+s+2}{(s+1)(s^2+2s+2)}$. We know that this may be written in the form

$$\frac{N(s)}{D(s)} = \frac{s^2+s+2}{(s+1)(s^2+2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+2}. \qquad \text{(A.2)}$$

The first coefficient is easy to obtain, and is given by

$$A = (s+1)\frac{s^2+s+2}{(s+1)(s^2+2s+2)}\bigg|_{s=-1} = \frac{s^2+s+2}{s^2+2s+2}\bigg|_{s=-1} = \frac{1-1+2}{(1-2+2)} = 2.$$

The other two coefficients are not as easy to obtain, however, we must have the right side of (A.2) equal to the left side for all values of s. In particular, if we let s be some convenient value that is not at one of the poles, the information may be useful. For example, if $s = 0$, then

$$\frac{N(0)}{D(0)} = \frac{0^2+0+2}{(0+1)(0^2+2(0)+2)} = \frac{A}{0+1} + \frac{B(0)+C}{0^2+2(0)+2} \quad \Rightarrow \quad \frac{2}{2} = A + \frac{C}{2}.$$

Thus, $C = 2(1-A) = 2(1-2) = -2$. Another convenient value may be $s = 1$, in which case

$$\frac{N(1)}{D(1)} = \frac{1^2+1+2}{(1+1)(1^2+2(1)+2)} = \frac{A}{1+1} + \frac{B(1)+C}{1^2+2(1)+2} \quad \Rightarrow \quad \frac{4}{10} = \frac{A}{2} + \frac{B+C}{5}$$

and $B = \frac{1}{2}(4-5A) - C = -3+2 = -1$. Thus,

$$\frac{N(s)}{D(s)} = \frac{s^2+s+2}{(s+1)(s^2+2s+2)} = \frac{2}{s+1} + \frac{-(s+2)}{s^2+2s+2}.$$

Likewise, we could solve for B independently of C if we evaluate $sN(s)/D(s)$ at $s = \infty$. Note that we multiply by s so that we will get something other than 0. Thus,

$$\left.\frac{sN(s)}{D(s)}\right|_{s=\infty} = \lim_{s\to\infty} \frac{s(s^2+s+2)}{(s+1)(s^2+2s+2)} = 1 = \lim_{s\to\infty} \frac{sA}{s+1} + \frac{s(Bs+C)}{s^2+2s+2} = \frac{A}{1} + \frac{B}{1}$$

and thus, $B = 1 - A = 1 - 2 = -1$, giving the same result we already obtained.

If we have access to MATLAB, then we may use the command **residue**. The following scripts can be used to compute the partial fraction expansion for the previous examples

```
num=[1 1];
den=conv([1 2],[1 2]);
den=conv([1 2],den);
[r,p]=residue(num,den);
A1=r(1),A2=r(2),A3=r(3)
```

```
num=[1 10 4];
den=conv([1 2],[1 2]);
den=conv([1 2],den);
den=conv([1 3],den);
[r,p]=residue(num,den);
A1=r(1),A2=r(2),A3=r(3),A4=r(4)
```

```
num=[1 1 2];
den=conv([1 1],[1 2 2]);
[r,p]=residue(num,den)
b1=r(1);b1c=r(2);p1=-p(2);
p1c=-p(1);
A=r(3),B=b1+b1c,C=b1*p1+b1c*p1c
```

Note that the **conv** command is used to "multiply" the two polynomials together. We note also that in the last example, the **residue** function factors the polynomial completely and returns complex coefficients with the complex poles. Thus, we need to combine them to obtain the correct coefficients.

C. Terminology

There are many terms connected with polynomials. Since polynomials play an important role in control systems, it is helpful to know these. In addition, since transfer functions are ratios of polynomials, it is helpful to know properties of rational functions. Here we list some words and their definitions relating to polynomials

Degree: the degree of a polynomial is the order of the polynomial, written deg(•). (e.g., if $D(s) = s^3 + s + 2$, then $\deg(D) = 3$).

Monic: a polynomial is a monic polynomial if the leading coefficient is 1. (e.g, if $D(s) = s^3 + s + 2$, and $C(s) = 2s^3 + s + 2$, then $D(s)$ is monic and $C(s)$ is not)

Coprime: two polynomials are said to be coprime if they have no common factors. (e.g., $N(z) = (z+1)(z-1)$, and $D_1(z) = (z = 2)(z-1)$ are not coprime, while $N(z)$ and $D_2(z) = (z+2)(z-2)$ are coprime). If two polynomials correspond to the numerator and denominator of a rational function, this is the same as saying there are no pole-zero cancellations.

For rational functions, we have the additional definitions

Relative Degree: the relative degree of a rational function, $N(s)/D(s)$, is the degree of the denominator minus the degree of the numerator. (i.e., rel deg = $\deg(D) - \deg(N)$)

Proper: A rational function $N(s)/D(s)$ is proper if the relative degree is non negative. (i.e. if $\deg(D) \geq \deg(N)$) (possibly zero)

Strictly Proper: A rational function $N(s)/D(s)$ is proper if the relative degree is postive. (i.e. if $\deg(D) > \deg(N)$)

D. The Diophantine Equation

The polynomial equation

$$N_1(x)N_2(x) + D_1(x)D_2(x) = P(x) \tag{A.3}$$

plays an important role in control systems. This equation is called the *Diophantine equation*, and is sometimes referred to as the *Bezout Identity*. This is a linear equation in the (unknown) polynomials $N_2(x)$ and $D_2(x)$, for known polynomials $N_1(x)$, $D_1(x)$ and $P(x)$. Here, we consider them to be polynomials in the variable x. If we desire to apply this in continuous-time, we let $x = s$, and if in discrete-time, we let $x = z$.

If $N_1(x)$ and $D_1(x)$ are coprime, there will always be a solution to this equation. Indeed, there are an infinite number of solutions. If $N_2(x)$ and $D_2(x)$ are solutions of (A.3), then for

$$\overline{N}_2(x) = N_2(x) + D_1(x)Q(x), \quad \overline{D}_2(x) = D_2(x) - N_1(x)Q(x) \tag{A.4}$$

the polynomials $\overline{N}_2(x)$ and $\overline{D}_2(x)$ are also solutions of (A.3) for any polynomial Q. This follows from

$$\begin{aligned} N_1\overline{N}_2 + D_1\overline{D}_2 &= N_1\left(N_2 + D_1Q\right) + D_1\left(D_2 - N_1Q\right) \\ &= N_1N_2 + D_1D_2 + N_1D_1Q - N_1D_1Q \\ &= P(x) \end{aligned}$$

Since there are many solutions, we must concern ourselves with solutions that are appropriate for our desired context. This will place constraints upon the solution of the equation (A.3). If equation (A.3) had a unique solution, we could have trouble because its application may not be valid in a particular design context. Some constraints that may be important to us are

- the polynomial $D_2(x)$ must be monic. This often occurs because of the solution of the equation yields polynomials that are the numerator and denominator of a controller. In general, we may take the denominator to be monic if we don't take the numerator to be monic.
- the degree of $D_2(x)$ must be greater than or equal to the degree of $N_2(x)$. This type of constraint may arise because of causality or realizability conditions.
- the degree of $D_2(x)$ must be greater than or equal to the degree of $N_2(x)$ plus some number that depends upon $N_1(x)$, $D_1(x)$ and $P(x)$. This type of constraint may arise because of causality conditions, or conditions that arise in a larger context of the overall design problem using the Diophantine equation.
- the degrees of $D_2(x)$ and $N_2(x)$ may depend upon $N_1(x)$, $D_1(x)$ and $P(x)$ as well as that of other polynomials.
- the polynomials $D_2(x)$ and $N_2(x)$ may be required to have certain roots. This may arise in a control setting where a certain property of the system can only be satisfied if certain roots exist in a controller.
- if $D_2(x)$ and $N_2(x)$ are continuous-time polynomials, the relative degree may guarantee a certain rolloff of the overall response
- if $D_2(x)$ and $N_2(x)$ are discrete-time polynomials, the relative degree may indicate the amount of delay in the controller.

D.1 An Example

Suppose we seek a solution to the polynomial equation

$$N_1(x)N_2(x) + D_1(x)D_2(x) = P(x)$$

where

$$N_1(x) = 3x^2 + x - 1, \quad D_1(x) = x^2 + 2x + 2, \quad P(x) = x^4 + 4x^3 + 6x^2 + 4x + 1$$

such that $\deg\big(D_2(x)\big) \geq \deg\big(N_2(x)\big)+1$, and such that $D_2(x)$ is monic. In this case, we consider

$$N_2(x) = a_1 x + a_0, \qquad D_2(x) = x^2 + b_1 x + b_0 .$$

This satisfies the restrictions on the two polynomials. Thus, we must have

$$\begin{aligned} x^4 + 4x^3 + 6x^2 + 4x + 1 &= (3x^2 + x - 1)\big(a_1 x + a_0\big) + (x^2 + 2x + 2)\Big(x^2 + b_1 x + b_0\Big) \\ &= \big(1\big)x^4 + \big(3a_1 + 2 + b_1\big)x^3 + \big(3a_0 + a_1 + b_0 + 2b_1 + 2\big)x^2 \\ &\quad + \big(a_0 - a_1 + 2b_1 + 2b_0\big)x + \big(-a_0 + 2b_0\big) \end{aligned}$$

Equating coefficients, we find

$$s^4 : 1 = 1$$

$$s^3 : 4 = 3a_1 + 2 + b_1 \quad \Rightarrow \quad 3a_1 + b_1 = 2$$

$$s^2 : 6 = 3a_0 + a_1 + b_0 + 2b_1 + 2 \quad \Rightarrow \quad 3a_0 + a_1 + b_0 + 2b_1 = 4$$

$$s^1 : 4 = a_0 - a_1 + 2b_1 + 2b_0 \quad \Rightarrow \quad a_0 - a_1 + 2b_1 + 2b_0 = 4$$

$$s^0 : 1 = -a_0 + 2b_0 \quad \Rightarrow \quad -a_0 + 2b_0 = 1$$

The first equation gives no new information. Thus, we have four equations, and four unknowns. This equation can thus be written (and solved) in the matrix form

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & 1 & 1 & 3 \\ 2 & 2 & -1 & 1 \\ 0 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_0 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} b_1 \\ b_0 \\ a_1 \\ a_0 \end{bmatrix} = \tfrac{1}{29} \begin{bmatrix} 43 \\ 16 \\ 5 \\ 3 \end{bmatrix}.$$

Thus, we obtained the polynomial solutions

$$N_2(x) = \tfrac{1}{29}\big(5x + 3\big), \qquad D_2(x) = x^2 + \tfrac{43}{29}x + \tfrac{16}{29}.$$

This example shows some important properties of the solution of the Diophantine equation. First, if there are degree constraints on the polynomials, such as we had there, then the number of possible solutions decreases greatly. Indeed, in the previous example, if we considered, for $Q = 1$,

$$\overline{N}_2(x) = N_2(x) + D_1(x)Q(x) = \left(\tfrac{1}{29}\right)(5x+3) + (x^2 + 2x + 2)(1) = x^2 + \tfrac{63}{29}x + \tfrac{61}{29}$$

$$\overline{D}_2(x) = D_2(x) - N_1(x)Q(x) = \left(x^2 + \tfrac{43}{29}x + \tfrac{16}{29}\right) - (3x^2 + x - 1)(1) = -2x^2 + \tfrac{14}{29}x + \tfrac{45}{29}$$

these polynomials are solutions of the Diophantine equation, but do not satisfy the degree condition

$$\deg\left(\overline{D}_2(x)\right) \geq \deg\left(\overline{N}_2(x)\right) + 1.$$

Indeed, by applying very tough constraints, it may not be possible to obtain a solution.

The second property illustrated in the example is that the polynomial problem can be translated into a linear algebra problem. The solution of linear equations is discussed in Appendix C.

The third important property is that if there are constraints on the polynomials, then the polynomial $P(x)$ must be of appropriate order. Indeed, we often must have

$$\deg(P) = \max\left\{\deg(N_1 N_2), \deg(D_1 D_2)\right\}. \tag{A.5}$$

This condition guarantees that the right and left hand sides of (A.3) are the same order of polynomial. This condition can be relaxed under very special circumstances. Along with this equation, we generally must have

$$\deg(N_2) + \deg(D_2) > \deg(P). \tag{A.6}$$

This equation arises because the underlying linear algebra problem will generally not be consistent if the set of equations is overdetermined.

D.2 The General Solution

When there are an infinite number of solutions to a problem, it is often helpful to reduce the possibilities. These possibilities are automatically reduced when there are constraints on the degrees of the polynomials. Here, we consider the polynomials to have degrees given by the fixed numbers

$$\begin{aligned} n_1 &= \deg(N_1), \quad n_2 = \deg(N_2), \\ d_1 &= \deg(D_1), \quad d_2 = \deg(D_2), \quad p_1 = \deg(P) \end{aligned} \tag{A.7}$$

This is the general solution, because we haven't specified the values. As suggested in (A.5), in order to have a solution, we may need to have

$$p_1 = \max\{(n_1 + n_2), (d_1 + d_2)\}. \tag{A.8}$$

This equation basically says that the right and left hand sides of (A.3) are the same order of polynomial. This condition may not be necessary if

$$n_1 + n_2 = d_1 + d_2 > p_1 \tag{A.9}$$

In this case, the first $n_1 + n_2 - p_1$ coefficients of the left hand side of (A.3) must all be zero. For now we will assume that (A.8) holds.

In general, we must also have

$$n_2 + d_2 + 1 \geq p_1. \tag{A.10}$$

This condition arises because otherwise, the underlying linear algebra problem will be inconsistent. This will be seen in the following development.

In order to solve the problem in the previous example, we transformed the polynomial problem into a linear algebra problem. To do this in the general case, note that for

$$N_1(x)N_2(x) + D_1(x)D_2(x) = P(x) \tag{A.11}$$

we explicitly write our given polynomials as

$$N_1(x) = a_{n_1}x^{n_1} + a_{n_1-1}x^{n_1-1} + \cdots + a_1x + a_0 \tag{A.12}$$

$$D_1(x) = x^{d_1} + b_{d_1-1}x^{d_1-1} + \cdots + b_1x + b_0 \tag{A.13}$$

$$P(x) = x^{p_1} + c_{p_1-1}x^{p_1-1} + \cdots + c_1x + c_0 \tag{A.14}$$

and we seek to obtain the polynomials

$$N_2(x) = \alpha_{n_2}x^{n_2} + \alpha_{n_2-1}x^{n_2-1} + \cdots + \alpha_1x + \alpha_0 \tag{A.15}$$

$$D_2(x) = x^{d_2} + \beta_{d_2-1}x^{d_2-1} + \cdots + \beta_1x + \beta_0 \tag{A.16}$$

We have assumed, without loss of generality, that $P(x)$ is monic. This we can do because we may divide (A.11) by the leading coefficient of $P(x)$ (which must be nonzero). We have also assumed that $D_1(x)$ and $D_2(x)$ are monic. If $p_1 = d_1 + d_2 > n_1 + n_2$, then the leading coefficient of the left and right hand sides is equal to 1 (as in the example). If we have

$p_1 = n_1 + n_2 + 1 \geq d_1 + d_2$, then we are not inconsistent since the leading coefficient of $N_2(x)$ can be used to ensure the equality of the right and left hand sides of (A.11).

In order to transform the polynomial equation (A.11) into a linear algebra problem, we expand the products of the polynomials in (A.11), equate coefficients of the various powers of x, and stack the equations into a matrix form. We have two cases that we can consider.

Case 1

In this case, we have $p_1 = d_1 + d_2 > n_1 + n_2$. This will occur most often because the Diophantine equation is associated with a control system. By causality, the denominator must be of greater order than or equal the numerator. In this case, we obtain the matrix equation

$$
\begin{array}{c}
p_1 = d_1 + d_2 \text{ rows} \\
\left[\begin{array}{cccc|cccc}
1 & 0 & \cdots & 0 & & & & \\
b_{d_1-1} & 1 & & \vdots & & & & \\
\vdots & b_{d_1-1} & \ddots & 0 & & & & \\
b_1 & \vdots & & 1 & & & & \\
b_0 & b_1 & & b_{d_1-1} & a_{n_1} & & 0 & \\
0 & b_0 & \ddots & \vdots & \vdots & \ddots & & \\
\vdots & & \ddots & b_1 & a_1 & & a_{n_1} & \\
0 & \cdots & 0 & b_0 & a_0 & \ddots & \vdots & \\
 & & & & 0 & & a_1 & \\
 & & & & & \ddots & & \\
 & & & & & & a_0 &
\end{array}\right] \\
\underbrace{\qquad\qquad\qquad}_{d_2 \text{ columns}} \quad \underbrace{\qquad\qquad\qquad}_{n_2+1 \text{ columns}}
\end{array}
\left[\begin{array}{c}
\beta_{d_2-1} \\ \vdots \\ \beta_1 \\ \beta_0 \\ \alpha_{n_2} \\ \vdots \\ \alpha_1 \\ \alpha_0
\end{array}\right]
=
\left[\begin{array}{c}
c_{p_1-1} - b_{d_1-1} \\ c_{p_1-2} - b_{d_1-2} \\ \vdots \\ c_{p_1-d_1+1} - b_1 \\ c_{p_1-d_1} - b_0 \\ c_{p_1-d_1-1} \\ \vdots \\ c_0
\end{array}\right]
\qquad \text{(A.17)}
$$

$r_1 = p_1 - n_1 - n_2 - 1$, $0_{(r_1)\times(n_2+1)}$

or more compactly as

$$RY = Z \qquad \text{(A.18)}$$

where the matrix $R \in \mathbb{R}^{p_1 \times (d_2 + n_2 + 1)}$ is a special matrix called the *Resultant matrix*. This matrix is sometimes referred to as the *Resolvant matrix*, the *Sylvester matrix* or the *Eliminant matrix*. It is a matrix that occurs often in applied mathematics. It has the property that it is full rank if and only if the polynomials $N_1(x)$ and $D_1(x)$ have no common factors. A solution to this equation exists if

$$\text{rank}(R) = p_1. \qquad \text{(A.19)}$$

This requires that

$$d_1 \leq n_2 + 1. \qquad \text{(A.20)}$$

This arises because the number of columns of R must be greater than or equal the number of rows . This system of equations must not be overdetermined or else it will be inconsistent. If (A.20) is satisfied, then (A.18) has the least-squares solution

$$Y = R^{\dagger} Z \tag{A.21}$$

and the general solution

$$Y = R^{\dagger} Z + (I - R^{\dagger} R) W \tag{A.22}$$

for any $W \in \mathbb{R}^{d_2 + n_2 + 1}$. If $d_1 = n_2 + 1$, then $R^{\dagger} = R^{-1}$, and there is a unique solution to (A.21).

<u>Example A.10</u>: Given $N_1(x) = 6x^3 + 4x^2 + 2x + 1$, $D_1(x) = x^3 + 7x^2 + 5x + 3$ and $P(x) = x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$. We wish to find a solution such that $\deg(D_2) > \deg(N_2)$. Note that we have $n_1 = 3, d_1 = 3, p_1 = 7$. In this problem, we must have $d_1 + d_2 = p_1$, and so we must have $d_2 = 4$. In order for this problem to have a solution, we must have $3 = d_1 \le n_2 + 1 < d_2 + 1 = 5$, or $2 \le n_2 < 4$. Thus, we may have n_2 be either 2 or 3. If we select $n_2 = 2$, then we obtain the equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 1 & 0 & 0 & 6 & 0 & 0 \\ 5 & 7 & 1 & 0 & 4 & 6 & 0 \\ 3 & 5 & 7 & 1 & 2 & 4 & 6 \\ 0 & 3 & 5 & 7 & 1 & 2 & 4 \\ 0 & 0 & 3 & 5 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} 1-7 \\ 1-5 \\ 1-3 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} -6 \\ -4.522 \\ -2.5402 \\ -0.9796 \\ 7.087 \\ 5.6411 \\ 3.9388 \end{bmatrix}$$

which is a unique solution:

$$N_2(x) = 7.087x^2 + 5.6411x + 3.9388$$

$$D_2(x) = x^4 - 6x^3 - 4.522x^2 - 2.5402x - 0.9796$$

If we select $n_2 = 3$, then we obtain the equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 7 & 1 & 0 & 0 & 4 & 6 & 0 & 0 \\ 5 & 7 & 1 & 0 & 2 & 4 & 6 & 0 \\ 3 & 5 & 7 & 1 & 1 & 2 & 4 & 6 \\ 0 & 3 & 5 & 7 & 0 & 1 & 2 & 4 \\ 0 & 0 & 3 & 5 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \\ \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \\ -2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \\ \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} 0.3736 \\ -0.273 \\ -0.4157 \\ 0.0826 \\ -1.0623 \\ -0.3488 \\ 0.3297 \\ 0.7521 \end{bmatrix}$$

where we have the least-squares solution:

$$N_2(x) = -1.0623x^3 - 0.3488x^2 + 0.3297x + 0.7521$$

$$D_2(x) = x^4 + 0.3736x^3 - 0.273x^2 - 0.4157x + 0.0826$$

The two solutions are very different from each other. Note in the second case the Y is larger than the first because of the additional element α_3.

Case 2

We now have $p_1 = n_1 + n_2 \geq d_1 + d_2$. In this case, we obtain the matrix equation, $RY = Z$, as

$$\begin{array}{c} p_1+1 = n_1 + n_2 + 1 \text{ rows} \\ \\ \end{array} \left[\begin{array}{c} \underbrace{\begin{matrix} 0_{(p_1-d_1-d_2+1)\times d_2} \\ \begin{matrix} 1 & & & 0 \\ b_{d_1-1} & \ddots & & \\ \vdots & & 1 & \\ b_1 & & b_{d_1-1} & \\ b_0 & & \vdots & \\ & \ddots & b_1 & \\ 0 & & b_0 & \end{matrix} \end{matrix}}_{d_2 \text{ columns}} \quad \underbrace{\begin{matrix} a_{n_1} & 0 & \cdots & 0 \\ a_{n_1-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_1 & \ddots & \ddots & a_{n_1} \\ a_0 & \ddots & \ddots & a_{n_1-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1 \\ 0 & \cdots & 0 & a_0 \end{matrix}}_{n_2+1 \text{ columns}} \end{array} \right] \begin{bmatrix} \beta_{d_2-1} \\ \vdots \\ \beta_1 \\ \beta_0 \\ \alpha_{n_2} \\ \vdots \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} 1 \\ c_{p_1-1} \\ \vdots \\ (c_{r_2+2}) \\ (c_{r_2+1}) - 1 \\ c_{r_2} - b_{d_1-1} \\ \vdots \\ (c_{p_1-d_2}) - b_0 \\ c_{p_1-d_2-1} \\ \vdots \\ c_0 \end{bmatrix} \quad \text{(A.23)}$$

$$r_2 = p_1 - d_1 - d_2$$

where $R \in \mathbb{R}^{p_1 \times (d_2 + n_2 + 1)}$. This matrix also has the property that it is full rank if and only if the polynomials $N_1(x)$ and $D_1(x)$ have no common factors. A solution exists if rank$(R) = p_1$ which requires that

$$d_2 \geq n_1. \tag{A.24}$$

This also arises because the number of columns of R must be greater than or equal the number of rows . This system of equations must not be overdetermined or else it will be inconsistent. If (A.24) is satisfied, then (A.23) has the least-squares solution

$$Y = R^{\dagger} Z \tag{A.25}$$

and the general solution

$$Y = R^{\dagger} Z + (I - R^{\dagger} R) W \tag{A.26}$$

for any $W \in \mathbb{R}^{d_2 + n_2 + 1}$. If $d_2 = n_1$, then $R^{\dagger} = R^{-1}$, and there is a unique solution to (A.25).

<u>Example A.11</u>: Given $N_1(x) = 2x + 4$, $D_1(x) = x + 3$ and $P(x) = x^3 + x^2 + x + 1$. We wish to find a solution such that $2 \geq \deg(D_2) + 1 \geq \deg(N_2)$. Note that we have $n_1 = d_1 = 1$, $p_1 = 3$. In this problem, we must have $n_1 + n_2 + 1 = p_1$, and so we must have $n_2 = 2$. We also have $d_2 = 1$, Thus we obtain the equation

$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 1 & 0 & 4 & 2 \\ 3 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1-1 \\ 1-3 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \beta_0 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1/2 \\ -1 \\ 5/2 \end{bmatrix}$$

which gives the unique solution

$$N_2(x) = 1/2\, x^2 - x + 5/2, \quad D_2(x) = x - 3.$$

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